Dynamic Stability of Axially-Stiffened Imperfect Cylindrical Shells under Axial Step Loading

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The dynamic stability of an axially-stiffened cylinder under an axial impulsive load in the form of a step function is investigated. Donnell-Kármán-type nonlinear equations are derived for the initially imperfect stiffened cylinders including eccentricity of stiffeners. A buckling criterion is defined and buckling loads are computed for different imperfection parameters and positive and negative eccentricities for a cylinder with an axial stiffening system. The results indicate the high imperfection sensitivity of axially-stiffened cylinders.

Nomenclature

a_1, a_2, a_3, a_4	= deflection parameters
A_s, A_r	= cross-sectional area of stiffeners
b_s, b_r	= stiffener spacing
d_1, d_2	= imperfection parameters
E	= Young's modulus of shell and stiffener, assumed same
$E_s, E_r\,(e_s, e_r)$	= stiffener eccentricity (dimensionless)
f	= dimensionless stress function
G	= shear modulus of skin and stiffener
$\hat{g}$	= dimensionless circumferential inertia load
H	= shell thickness
I_s, I_r	= moment of inertia of stiffener about shell middle surface
J_s, J_r	= torsional constant of stiffener
L	= cylinder length
m, n	= wave numbers in axial and circumferential directions
$M_x, M_y, M_{xy}, M_{yx}\,(m_x, m_y, m_{xy}, m_{yx})$	= stress couples (dimensionless)
$N_x, N_y, N_{xy}\,(n_x, n_y, n_{xy})$	= stress resultants (dimensionless)
$\hat{n}$	= critical load, dimensionless
R	= radius of shell
$U, V, W\,(u, v, w)$	= middle surface displacements (dimensionless)
$\bar{W}\,(\bar{w})$	= initial imperfection of shell middle surface (dimensionless)
$X, Y\,(x, y)$	= axial and circumferential coordinates (dimensionless)
Z	= Batdorf parameter, L^2/RH
α_s, α_r	= axial stiffness parameter, $[(1-\nu^2)/h][(A_s/b_s), (A_r/b_r)]$
η_s, η_r	= flexural stiffness parameter, $(E/D)[(I_s/b_s), (I_r/b_r)]$
γ_s, γ_r	= torsional stiffness parameter, $[G/D(1-\nu)][(J_s/b_s), (J_r/b_r)]$
ν	= Poisson ratio of shell and stiffeners
τ, τ_1	= dimensionless times
s, r	= denote stringers (axial stiffeners) and rings
	= denotes partial differentiation

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Introduction

THE stability of stiffened cylinders under axial load is of fundamental interest in aircraft and missile design as the typical fuselage or booster is a stiffened cylinder. The investigations in the past two decades, though confined primarily to the static stability case, have revealed the principal differences between the behavior of stiffened and unstiffened cases. First, the stiffener location has considerable effect on the buckling load. For example, if the stiffeners are inside the shell the buckling load is almost half that obtained with stiffeners outside.[1,2] Second, and perhaps more important, the large number of test results on stiffened shells[1,3] seemed to indicate that general instability occurred very close to the predictions of classical linear theory, in contrast to the behavior of unstiffened shells where large deviations from the classical buckling was the rule. In Ref. 4, a summary of the available experimental results for stiffened shells was reported together with the tentative regions of validity of the linear theory. It was noted in Ref. 4 that, among the stiffened shells, the stringer-stiffened shells were most likely to deviate from the linear theory predictions and the strength degradation would be mainly due to their imperfection-sensitivity. Hutchinson and Amazigo[5] and Brush[6] noted such imperfection-sensitivity in stringer-stiffened shells and Singer et al.[7] in the most recent and comprehensive study, reached similar conclusions.

As was noted, all these investigations were addressed to the static stability of stiffened shells. The dynamic counterpart of the problem, becoming increasingly important with the advent of missile structures, has hardly been touched. To the authors' knowledge, Klosner and Franklin's[8] study is the only one in the literature that comes close to the objective of this paper. However their work is very preliminary and incomplete, mainly extending the work of Roth and Klosner[9] on the dynamic buckling of unstiffened cylinders.

This paper forms part of comprehensive theoretical study, initiated at Army Materials and Mechanics Research Center, to investigate the dynamic general instability of stiffened shells under a variety of impulsive loads, and deals with the case of an axially-stiffened (stringer-stiffened) cylinder under axial step loading.

Governing Equations

The nonlinear equations of motion pertinent to the problem are based on the assumptions commonly used in a Donnell-type formulation, valid for moderate length cylinders. Additional assumptions for stiffeners are that: a) they are close enough so that they can be "smeared" out in calculating the effective rigidities; b) they are essentially beamlike elements carrying no shear; and c) the entire shell including the stiffeners is activated in the buckling; that is, local instability failures are ruled out.

The derivation is similar to that of Baruch and Singer[1] except that we now include nonlinearity in strain-displacement relationship and also initial imperfections.

Figure 1 shows the cylindrical shell with the stiffener geometry and the coordinate system. With the normal to the shell surface taken as positive inward, positive (negative) values of $\bar{E}_s$, $\bar{E}_r$ indicate inside (outside) location of the stiffeners with respect to the shell.

The unstiffened shell is characterized by three parameters L, R, and H (length, radius and the thickness of shell) which can be combined into the so-called Batdorf parameter $Z = L^2/RH$, the ranges whose values are used to describe a "short" or a "moderate-length" cylinder. For practical thin shells Z is in the range 10^2–10^4.

The effective increases in the area of section, the section area moment, and the section torsional stiffness of the shell due to the presence of stiffeners are best characterized by nondimensional parameters α, η and γ.

In order to derive the basic equations we introduce the following nondimensional quantities. All lengths (including X, Y the coordinate distances) are normalized with respect to R the shell radius. Thus we write

$$u,\, v,\, w,\, \bar{w},\, x,\, y,\, \bar{e}_s,\, \bar{e}_r = [U,\, V,\, W,\, \bar{W},\, X,\, Y,\, \bar{E}_s,\, \bar{E}_r]/R$$

Further nondimensional quantities may be defined from the stress and moment resultants as

$$n_x, n_y, n_{xy}, \bar{n}_x = [N_x, N_y, N_{xy}, \bar{N}_x]/B$$
$$m_x, m_y, m_{xy}, m_{yx} = [M_x, M_y, M_{xy}, M_{yx}]/D$$

where

$$B = Eh/(1-\nu^2) \qquad D = Eh^3/12(1-\nu^2)$$

Strain-Displacement Relationship

The assumed nonlinear strain-displacement relationships consistent with a Donnell-type derivation are

$$\varepsilon_x^{\,0} = u_{,x} + \tfrac{1}{2}w_{,x}^{\,2} + w_{,x}\bar{w}_{,x}$$
$$\varepsilon_y^{\,0} = v_{,y} - w + \tfrac{1}{2}w_{,y}^{\,2} + w_{,y}\bar{w}_{,y} \tag{1}$$

$$2\varepsilon_{xy}^{\,0} = u_{,y} + v_{,x} + w_{,x}w_{,y} + w_{,x}\bar{w}_{,y} + \bar{w}_{,x}w_{,y}$$
$$R\kappa_x = w_{,xx} \qquad R\kappa_y = w_{,yy} \qquad R\kappa_{xy} = w_{,xy} \tag{2}$$

In Eqs. (1) $\bar{w}$ represents the dimensionless initial imperfection.

Stress-Resultant-Strain Relationship

The stress-resultants are obtained by integrating over the thickness of the shell and the effective thickness of the stiffeners and are given[1] as

$$\begin{bmatrix} n_x \\ n_y \\ n_{xy} \\ m_x \\ m_y \\ m_{xy} \\ m_{yx} \end{bmatrix} = \left[\begin{array}{ccc|cccc} (1+\alpha_s) & \nu & 0 & -\bar{e}_s R\alpha_s & 0 & 0 & 0 \\ \nu & (1+\alpha_r) & 0 & 0 & -\bar{e}_r R\alpha_r & 0 & 0 \\ 0 & 0 & (1-\nu) & 0 & 0 & 0 & 0 \\ & M_1 & & & M_2 & & \\ \hline -\bar{e}_s\beta_s & 0 & 0 & (1+\eta_s) & \nu & 0 & 0 \\ 0 & -\bar{e}_r\beta_r & 0 & \nu & (1+\eta_r) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-\nu)(1+\gamma_s) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-\nu)(1+\gamma_r) \\ & M_3 & & & & M_4 & \end{array}\right] \begin{bmatrix} \varepsilon_x^{\,0} \\ \varepsilon_y^{\,0} \\ \varepsilon_{xy}^{\,0} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \kappa_{yx} \end{bmatrix} \tag{3}$$

The nonzero submatrices M_2 and M_3 in Eq. (3) provide the coupling effects of the stiffener eccentricity; also β_s (β_r) in Eq. (3) is not a new parameter but is related to α_s (α_r) through $D\beta_s = B\alpha_s$.

Of the various stiffness parameters, it is convenient to regard $\alpha_s (\alpha_r)$, $\eta_s (\eta_r)$ as independent parameters, though they are, strictly, related through the geometry. The torsional stiffness $\gamma_s (\gamma_r)$ can be expressed in terms of α_s for simple geometric shapes. For example, with a rectangular section and a depth greater than its width, one can write

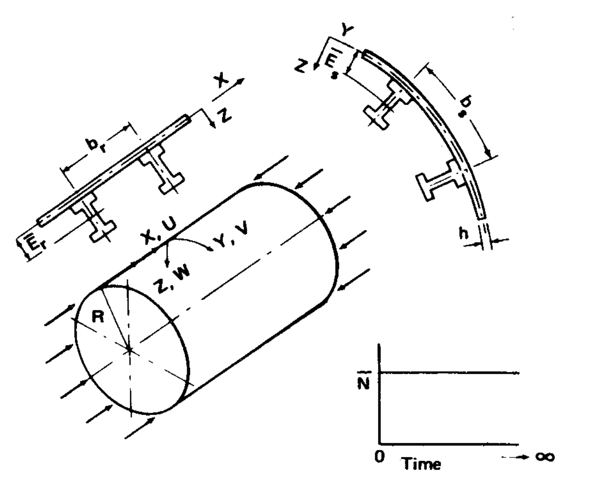

Fig. 1 Shell geometry and loading.

$$\gamma_s = 2\alpha_s k_s^{\,2}/(1-\nu^2)$$

where k_s = width of stiffener/shell thickness.

Assigning a convenient value for k_s ($\equiv 1$ in the present paper) γ_s is expressed entirely in terms of α_s.

Equations of Motion

The equations of motion are derived from the Hamilton Principle, which requires that the variation of a functional of the Hamiltonian potential over an arbitrary interval of time (t_0, t_1) should be zero. That is

$$\delta \int_{t_0}^{t} L\, dt = 0 \tag{4}$$

where L = kinetic energy (KE)—internal strain energy (IE)+ external work done (W_e)

$$\mathrm{KE} = \tfrac{1}{2}\rho H \textstyle\int_S (\dot{U}\dot{U} + \dot{V}\dot{V} + \dot{W}\dot{W})\, dX\, dY$$

$$\mathrm{IE} = \tfrac{1}{2}\textstyle\int_S [N_x\varepsilon_x^{\,0} + N_y e_y^{\,0} + 2N_{xy}\varepsilon_{xy}^{\,0} + M_x\kappa_x + M_y\kappa_y + (M_{xy}+M_{yx})\kappa_{xy}]\, dX\, dY$$

$$W_e = \tfrac{1}{2}\textstyle\int_S \bar{N}_x U_{,x}\, dX\, dY$$

S refers to the shell surface and $\bar{N}_x$ is the applied external load.

Upon substituting for the various terms in Eq. (4) from Eqs. (1–3), we obtain the following as the equations of motion:

$$n_{x,x} + n_{xy,y} = n_{xy,x} + n_{y,y} = 0 \tag{5a}$$

$$-(1/12)(H^2/R^2)L_3(w) + \alpha_s\bar{e}_s\varepsilon_{x,xx}^{\,0} + \alpha_r\bar{e}_r\varepsilon_{y,yy}^{\,0} + n_x w_{,xx}{}^{*} + 2n_{xy}w_{,xy}{}^{*} + n_y(w_{,xy}{}^{*}+1) = \partial^2 w/\partial\tau_1^{\,2} \tag{5b}$$

where L_3 is a linear differential operator defined by

$$L_3(\;) = (1+\eta_s)(\;)_{,xxxx} + 2\{1-(\alpha_s+\alpha_r)/(1+\nu)\}(\;)_{,xxyy} + (1+\eta_r)(\;)_{,yyyy}$$

In deriving Eqs. (5a) and (5b) the longitudinal inertia terms have been ignored.

It is readily verified that Eq. (5a) is identically satisfied by the introduction of a stress function f such that

$$n_x = f_{,yy} \qquad n_y = f_{,xx} \quad \text{and} \quad n_{xy} = -f_{,xy}$$

Equation (5b) will then be a single equation in terms of w and f. In order to have a second independent equation involving f and w we turn to Eqs. (1) which yield the following compatibility condition:

$$\varepsilon_{x,yy}{}^0 + \varepsilon_{y,xx}{}^0 - 2\varepsilon_{xy,xy}{}^0 = w_{,xy}{}^2 - w_{,xx}w_{,yy} - w_{,xx} + 2w_{,xy}\bar{w}_{,xy} - w_{,xx}\bar{w}_{,yy} - w_{,yy}\bar{w}_{,xx} \qquad (6)$$

Making use of f and the stress-strain relationships, Eq. (3), we can rewrite Eqs. (6) and (5b) as follows:

$$L_1 f - L_2 w = A_{rs}[w_{,xy}{}^2 - w_{,xx}w_{,yy} - w_{,xx} + 2w_{,xy}\bar{w}_{,xy} - w_{,xx}\bar{w}_{,yy} - w_{,yy}\bar{w}_{,xx}] \qquad (7)$$

and

$$-(1/12)(H^2/R^2)L_3 w - (1/A_{rs})[L_2 f - L_4 w] + f_{,xx}(w_{,yy}{}^* + 1) + f_{,yy}w_{,xx}{}^* - 2f_{,xy}w_{,xy}{}^* = \partial^2 w/\partial\tau^2 \qquad (8)$$

where the linear operators $L_1 L_2 L_3$ and L_4 are given by

$$L_1\Phi = (1+\alpha_s)\Phi_{,xxxx} + 2[1 + (\alpha_s + \alpha_r + \alpha_s\alpha_r)/(1-\nu)]\Phi_{,xxyy} + (1+\alpha_r)\Phi_{,yyyy}$$

$$L_2\Phi = \nu\alpha_s\bar{e}_s\Phi_{,xxxx} - [(1+\alpha_s)\bar{e}_r\alpha_r + (1+\alpha_r)\bar{e}_s\alpha_s]\Phi_{,xxyy} + \nu\alpha_r\bar{e}_r\Phi_{,yyyy}$$

$$L_3\Phi = (1+\eta_s)\Phi_{,xxxx} + 2[1 - (\alpha_s + \alpha_r)/(1+\nu)]\Phi_{,xxyy} + (1+\eta_r)\times\Phi_{,yyyy}$$

$$L_4\Phi = (\alpha_s\bar{e}_s)^2(1+\alpha_r)\Phi_{,xxxx} - 2\nu\alpha_s\alpha_r\bar{e}_r\bar{e}_s\Phi_{,xxyy} + (\alpha_r\bar{e}_r)^2(1+\alpha_s)\Phi_{,yyyy}$$

and

$$\tau = \tau_1[(1+\alpha_s)(1+\alpha_s) - \nu^2]^{1/2}$$

$$A_{rs} = A_{sr} = (1+\alpha_r)(1+\alpha_s) - \nu^2$$

Equations (7) and (8) are the governing equations of the problem.

These equations on the one hand reduce to the dynamic equations of Roth and Klosner[9] by setting all stiffening terms to zero; and on the other, by discarding eccentricity terms, reduce to those of Klosner and Franklin.[8] Linearization, and ignoring inertia terms, yields the linearized static stability equations used by Baruch and Singer.[1]

Instead of attempting to solve Eqs. (7) and (8) directly, we use a four-parameter modal approximation for w, which represents the diamond-shaped buckles familiar in static stability studies. This four-parameter modal which ignores the boundary conditions is similar to the one used by Roth and Klosner[9] in their study of the dynamic buckling of unstiffened cylinders.

Thus w is taken as

$$w = (H/R)[a_1(t)\cos m\bar{\pi}x\cos ny + a_2(t)\cos 2m\bar{\pi}x + a_3(t)\cos 2ny + a_4(t)] \qquad (9)$$

The initial imperfection $\bar{w}$ is taken as

$$\bar{w} = (H/R)[d_1\cos m\bar{\pi}x\cos ny + d_2\cos 2m\bar{\pi}x], \quad \bar{\pi} = \pi R/L \qquad (10)$$

Making use of Eqs. (9) and (10) in the compatibility relation, Eq. (6), we solve for f and obtain as a result

$$f = Q_1\cos 3m\bar{\pi}x\cos ny + Q_2\cos m\bar{\pi}x\cos 3ny + Q_3\cos 2m\bar{\pi}x\cos 2ny + Q_4\cos m\bar{\pi}x\cos ny + Q_5\cos m\bar{\pi}x + Q_6\cos 2ny - \tfrac{1}{2}\bar{n}y^2 + \tfrac{1}{2}\bar{g}x^2 \qquad (11)$$

where

$$Q_1 = -(2A_{rs}\beta^2/Z^2M_{rs})(a_1a_2 + a_2d_1 + a_1d_2)$$

$$Q_2 = -(2A_{rs}\beta^2/Z^2N_{rs})(a_1 + d_1)a_3$$

$$Q_3 = -(A_{rs}\beta^2/Z^2P_{rs})a_3(a_2 + d_2)$$

$$Q_4 = -\frac{2A_{rs}}{Z^2}\frac{\beta^2}{P_{rs}}\left(a_1a_2 + a_1a_3 - \frac{\alpha}{2}a_1 + a_2d_1 + a_1d_2 + a_3d_1 - \frac{B_{rs}}{2\beta A_{rs}}a_1\right)$$

$$Q_5 = -\frac{A_{rs}}{16Z^2}\frac{\beta^2}{(1+\alpha_s)}\left[\frac{a_1{}^2}{2} + a_1d_1 - 4\alpha a_2 - \frac{16\nu\alpha_s e_s}{\beta^2 A_{rs}}\alpha_2\right]$$

$$Q_6 = -\frac{A_{rs}}{16Z^2(1+\alpha_r)}\frac{1}{\beta^2}\left[\frac{a_1{}^2}{2} + a_1d_1 - 16\nu\alpha_r e_r\beta^2\frac{1}{A_{rs}}a_3\right]$$

and

$$M_{rs} = 81(1+\alpha_s) + 18[A_{rs}/(1-\nu) - \nu]\beta^2 + (1+\alpha_r)\beta^4$$

$$N_{rs} = (1+\alpha_s) + 18[A_{rs}/(1-\nu) - \nu]\beta^2 + 81(1+\alpha_r)\beta^4 \qquad (12)$$

$$P_{rs} = (1+\alpha_s) + 2[A_{rs}/(1-\nu) - \nu]\beta^2 + (1+\alpha_r)\beta^4$$

$$\alpha = Z/n^2 \qquad \beta = n/m\bar{\pi}$$

$$B_{rs} = \nu\alpha_s e_s - \{(1+\alpha_s)d_r e_r + (1+\alpha_r)d_s e_s\}\beta^2 + \nu\alpha_r e_r\beta^4;$$

$$e_r, e_s = (R/H)(\bar{e}_r, \bar{e}_s)$$

In Eq. (11) $\bar{n}$ is the average applied compressive load (nondimensional) per unit length and $\bar{g}$ is the average circumferential load per unit length arising out of the outward expansion of w. The quantity $\bar{g}$ being an unknown, we need another independent equation for determining $\bar{g}$. This is provided by the periodicity requirement on the circumferential displacement v.

Periodicity Condition

The requirement that the displacement v be single-valued, as we make a circuit around the cylinder, is expressed mathematically as

$$\int_0^{2\pi} v_{,y}\,dy = 0$$

or using the second of Eqs. (1), we have for the periodicity condition

$$\int_0^{2\pi}(\varepsilon_y{}^0 + w - \tfrac{1}{2}w_{,y}{}^2 - w_{,y}\bar{w}_{,y})\,dy = 0 \qquad (13)$$

By rewriting Eq. (13) in terms of f, w, and $\bar{w}$, and noticing that they already have periodic terms in y, we find that the requirement is satisfied by setting all the nonperiodic terms to zero. This results in the following:

$$a_4 + (1+\alpha_s)\frac{\bar{g}Z}{A_{rs}} + \nu\frac{Z\bar{n}}{A_{rs}} = \frac{1}{4\alpha}\left(\frac{a_1{}^2}{2} + a_1d_1 + 4a_3{}^2\right) \qquad (14)$$

Introducing further nondimensional quantities

$$\hat{n} = Z\bar{n}/A_{rs} \quad \text{and} \quad \hat{g} = Z\bar{g}/A_{rs}$$

we have from Eq. (14)

$$(1+\alpha_s)\hat{g} = \frac{1}{4\alpha}\left(\frac{a_1{}^2}{2} + a_1d_1 + 4a_3{}^2\right) - \nu\hat{n} - a_4 \qquad (15)$$

Separated Equations of Motion

Having the expression for f from Eq. (11), we substitute for f in Eq. (8) to obtain a single equation for w. This is solved, then, by using a Galerkin-type procedure to yield four ordinary nonlinear differential equations for the normal coordinates a_1–a_4.

These four equations together with Eq. (15) for $\hat{g}$ will constitute the set of equations for the unknowns a_1, a_2, a_3, a_4 and $\hat{g}$ in terms of the input parameters.

Thus, we rewrite Eq. (8) as

$$\chi(w) \equiv \partial^2 w/\partial\tau^2 + (1/12)1/(z)^2 L_3 w + (1/A_{rs})[L_2 f - L_4 w] + 2f_{,xy}w_{,xy}{}^* - f_{,xx}(w_{,yy}{}^* + 1) - f_{,yy}w_{,xx}{}^* = 0 \qquad (16)$$

Using Galerkin's procedure, we take

$$\hat{w} = \sum_{i=1}^{4}\hat{w}_i$$

with

$$\hat{w}_1 = \cos m\bar{\pi}x\cos ny, \quad \hat{w}_2 = \cos 2m\bar{\pi}x, \quad \hat{w}_3 = \cos 2ny, \quad \hat{w}_4 = 1$$

and set

$$\int_0^1\int_0^{2\pi}\chi(w)\hat{w}_i\,dx\,dy = 0 \qquad i = 1, 2, 3, 4 \qquad (17)$$

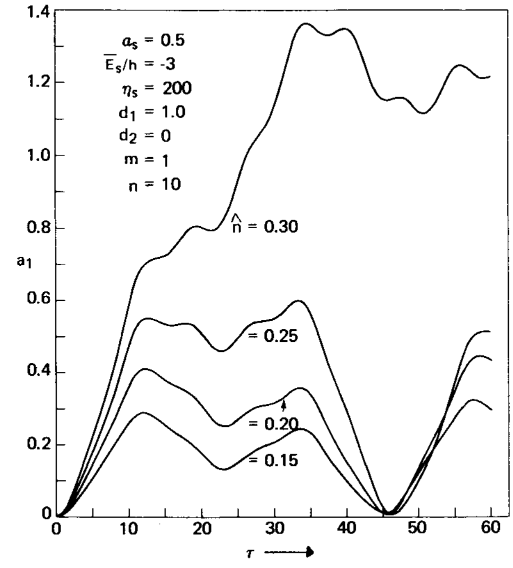

Fig. 2a Variation of a_1 with τ.

Carrying out the integration of Eq. (17), we obtain the following ordinary nonlinear differential equations for a:

$$\frac{1}{2}\ddot{a}_1+a_1\left[\frac{1}{24}\frac{1}{\alpha^2\beta^4}\frac{C_{rs}}{A_{rs}}-\frac{1}{2}\frac{1}{\alpha^2\beta^4}\frac{D_{rs}}{(A_{rs})^2}-\frac{1}{2}\frac{\hat{n}}{\alpha\beta^2}\right]-$$
$$\frac{1}{\alpha^2\beta^2}\frac{B_{rs}}{A_{rs}}\tilde{Q}_4+\frac{2}{\alpha^2}\tilde{Q}_1(a_2+d_2)+\frac{2}{\alpha^2}\tilde{Q}_2a_3+\frac{\hat{g}}{2\alpha}(a_1+d_1)+$$
$$\frac{2}{\alpha^2}\tilde{Q}_4\left(a_3+a_2+d_2-\frac{\alpha}{2}\right)+\frac{1}{16}\frac{1}{\alpha^2}\tilde{Q}_5(a_1+d_1)+$$
$$\frac{1}{16\alpha^2}\frac{1}{\beta^4}\tilde{Q}_6(a_1+d_1)=\frac{1}{2}\frac{1}{\alpha\beta^2}\hat{n}d_1 \qquad (18)$$

$$\ddot{a}_2+a_2\left[\frac{4}{3}\frac{1}{\alpha^2\beta^4}\frac{1+\eta_s}{A_{rs}}-\frac{16}{\alpha^2\beta^4}\frac{(e_sa_s)^2(1+\alpha_r)}{(A_{rs})^2}-\frac{4}{\alpha\beta^2}\hat{n}\right]-$$
$$\left(\frac{1}{\alpha^2\beta^2}\right)\frac{va_se_s}{A_{rs}}\tilde{Q}_5+\frac{2}{\alpha^2}\tilde{Q}_1(a_1+d_1)+\frac{8}{\alpha^2}\tilde{Q}_3a_3+\frac{2}{\alpha^2}\tilde{Q}_4(a_1+d_1)-$$
$$(1/4\alpha)\tilde{Q}_5=(4/\alpha\beta^2)\hat{n}\,d_2 \qquad (19)$$

$$\ddot{a}_3+a_3\left[\frac{4}{3\alpha^2}\frac{(1+\eta_r)}{A_{rs}}-\frac{16}{\alpha^2}\frac{(\alpha_re_r)^2(1+\alpha_s)}{(A_{rs})^2}\right]-$$
$$\frac{\beta^2}{\alpha^2}\frac{v\alpha_re_r}{A_{rs}}\tilde{Q}_6+\frac{2}{\alpha^2}\tilde{Q}_2(a_1+d_1)+\frac{2}{\alpha^2}\tilde{Q}_4(a_1+d_1)+$$
$$(8/\alpha^2)\tilde{Q}_3(a_2+d_2)+(4/\alpha)\hat{g}a_3=0 \qquad (20)$$
$$\ddot{a}_4=\hat{g} \qquad (21)$$

where

$$\tilde{Q}_1=(1/M_{rs})(a_1a_2+a_2d_1+a_1d_2)$$
$$\tilde{Q}_2=(1/N_{rs})\,a_3(a_1+d_1)$$
$$\tilde{Q}_3=(1/P_{rs})\,a_3(a_2+d_2)$$
$$\tilde{Q}_4=(1/P_{rs})[a_1a_2+a_1a_3-(\alpha/2)a_1+a_2d_1+a_1d_2+a_3d_1-(B_{rs}/2\beta^2A_{rs})a_1] \qquad (22)$$
$$\tilde{Q}_5=\frac{1}{1+\alpha_s}\left[\frac{a_1{}^2}{2}+a_1d_1-4\alpha a_2-\{16v\alpha_se_s/(\beta^2A_{rs})\}a_2\right]$$
$$\tilde{Q}_6=\frac{1}{1+\alpha_r}\left(\frac{a_1{}^2}{2}+a_1d_1-16v\alpha_re_r\beta^2a_3/A_{rs}\right)$$

B_{rs}, M_{rs}, N_{rs}, P_{rs} are the same as in Eq. (12) and

$$D_{rs}=(\alpha_se_s)^2(1+\alpha_r)-2v\alpha_sd_re_re_s\beta^2+(1+\alpha_s)(\alpha_re_r)^2\beta^4$$

In Eqs. (18–21) one can eliminate $\hat{g}$ by making use of Eq. (15). Eqs. (18–21) are a system of nonlinear differential equations which can be solved numerically with appropriate initial conditions for a_i and $\dot{a}_i$ to obtain a_i as function of τ for given values of input parameters.

Input Parameters

The input parameters include the stiffening parameters, α_s, α_r, η_r, η_s; the eccentricity parameters e_s, e_r; Batdorf parameter Z; imperfection parameters d_1 and d_2; load parameter $\hat{n}$; and the wave numbers m and n.

Since this paper deals only with axial stiffeners, α_r, η_r, and e_r are necessarily equal to zero. The range of values for the axial stiffness parameters has been chosen to conform to light, medium, and heavy stiffenings.

Of the remaining structural parameters, Z has been taken as 1000 and the imperfection parameter d_2 set to zero; d_1 has been varied between (0, 1).

The load $\hat{n}$ is assumed to be applied suddenly and kept thereon indefinitely; i.e., in the form of a step function.

Since for every integer value of m and n we can associate a critical load and the buckling load is the lowest of these, we have a wide range of m and n values.

Buckling Load: Theoretical Considerations

Buckling Criterion

For a given set of input parameters, Eqs. (18–21) can be integrated numerically, with appropriate initial conditions, to yield a_1–a_4 as function of τ. A Runge-Kutta scheme was used to integrate the system of equations and the convergence of the solution was verified by altering the step size.

A sample of the output of a_1–a_4 for several $\hat{n}$ values is given in Fig. 2 for a set of stiffening parameters. It is evident from Figs. 2a–2d that a_1 mode is the dominant mode. Also it is evident that two stable oscillatory response levels exist with a jump between these levels corresponding to a critical load.

The buckling criterion used here is similar to that of Roth and Klosner[9] based on the criterion derived by Budiansky and Roth,[10] where a jump in the peak amplitude of the deflection is associated with a critical load.

Strictly speaking the jump to be noted is that of the unit end shortening δ defined through

$$\delta=-\int_0^1 u_{,x}\,dx \qquad (23)$$

which can be written, after substituting for u in terms of f and w, as follows:

$$\delta Z=(1+\alpha_r)\hat{n}+v\hat{g}+\frac{1}{8\alpha\beta^2}\{a_1{}^2+8a_2{}^2+2a_1d_1+16a_2d_2\} \qquad (24)$$

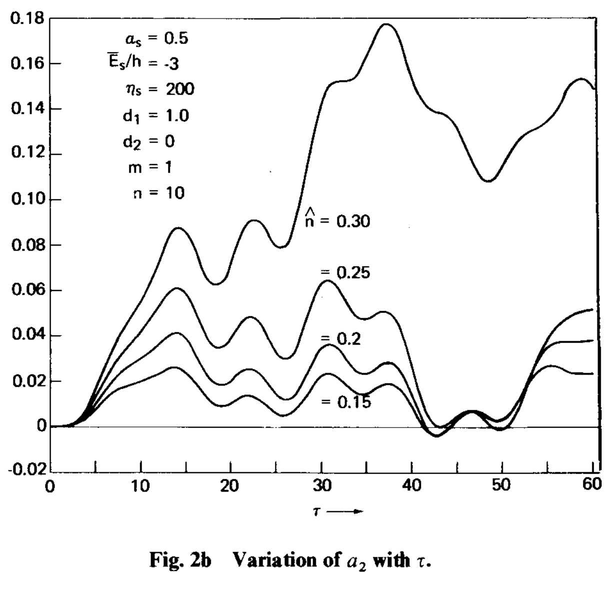

Fig. 2b Variation of a_2 with τ.

However, it was found that the jump in δ occurred at the same value as that of a_1 max, similar to the findings of Roth and Klosner.[9] Hence one could justifiably concentrate on the a_1 mode for determining the buckling load.

Modal Considerations

We have already remarked that in obtaining the buckling load we have to try a large number of m and n values to insure that we obtain the lowest critical values. This can be extremely time consuming even with high-speed digital computers. However this process is greatly simplified in the case of axially stiffened shells. Linear static buckling analysis[11] reveals that for certain minimum combinations of the stiffening parameters the axially stiffened shell always buckles in an asymmetric mode with $m = 1$; i.e., it buckles with many circumferential waves but only a single half wave axially.

Now we wish to establish the relevance of linear static value to the present problem by considering the limiting case of a perfect cylindrical shell (i.e., with $d_1 = d_2 \equiv 0$).

Perfect Cylinder Solution

Without loss of generality we can restrict our attention to the a_1 mode which, we have already said, is the dominant mode for determining the dynamic buckling load. Thus from Eq. (18) after setting $a_2 = a_3 = d_1 = d_2 = 0$, we obtain‡

$$\ddot{a}_1 - \frac{1}{\alpha\beta^2}(\hat{n} - G_{rs})a_1 + \frac{1}{16}\frac{1}{\alpha^2\beta^4}\left[\frac{\beta^4}{1+\alpha_s} + \frac{1}{1+\alpha_r}\right]a_1^{\,3} = 0 \quad (25)$$

where

$$G_{rs} = \frac{1}{A_{rs}}\frac{(m\pi)^2}{Z}\left[\frac{C_{rs}}{12} - \frac{1}{A_{rs}}\left(D_{rs} - \frac{B_{rs}^{\,2}}{P_{rs}}\right)\right] + \frac{Z}{(m\pi)^2}\frac{1}{P_{rs}} + \frac{2B_{rs}}{A_{rs}P_{rs}}$$

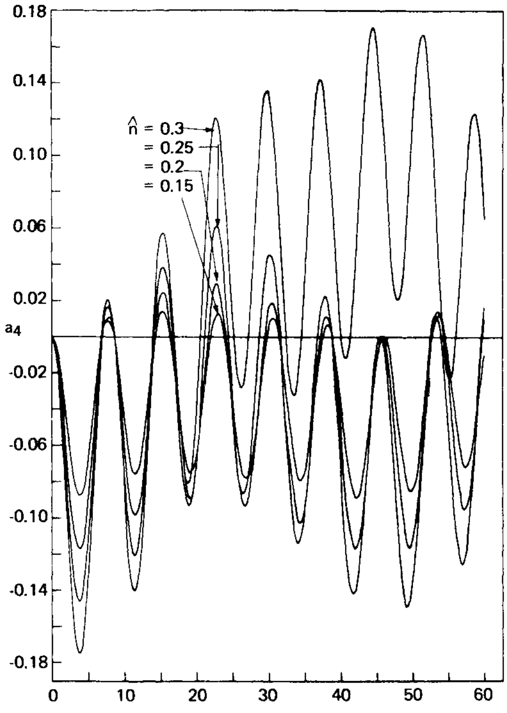

Fig. 2d Variation of a_4 with τ.

Although Eq. (25) is a nonlinear equation, it is apparent that the coefficient of the linear term alone governs the boundedness of the solution which is possible only for $\hat{n} \geq G_{rs}$. The nonlinearity only distorts the amplitude without affecting the basic stability. The minimum value for which the solution is bounded is obviously given by $\hat{n} = G_{rs}$. Now, this is the exact static linear buckling equation for the stiffened cylindrical shell under axial compression.[11]

Thus the static linear value provides the buckling solution for the dynamic stability of perfect cylinders. By letting all the stiffening parameters go to zero, we find that $\hat{n}$ reduces to the linear classical value of unstiffened cylinders. This fact, that the zero imperfection solution corresponds to the classical value, is noted in Ref. 9 also, though there it is presented as an a posteriori observation.

Having thus established the relationship of linear static value to the dynamic stability problem, it is only reasonable to assume that the mode shapes of the linear problem are relevant in the nonlinear problem of the imperfect cylinders. At least they provide useful starting values for maximum efficiency of computer time. We found that in all the cases we investigated the $m = 1$ mode persisted in the imperfect range, provided we selected the stiffening combinations which assured us of $m = 1$ asymmetric solution for the static linear problem.[11]

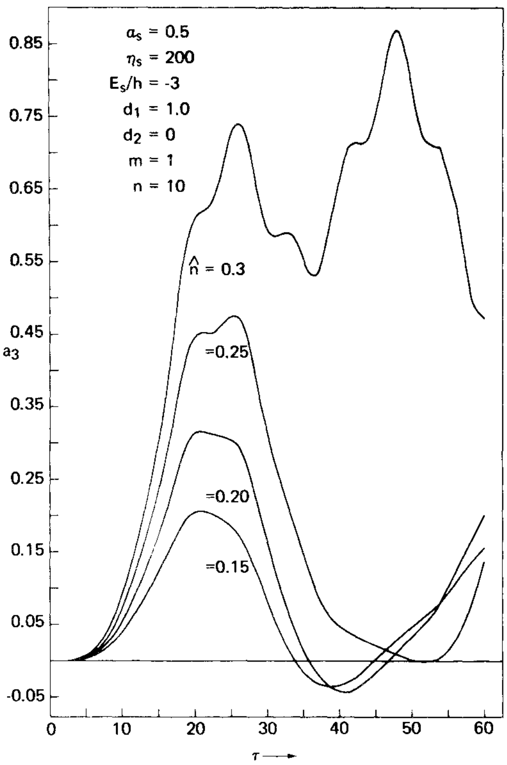

Fig. 2c Variation of a_3 with τ.

‡ We also set $\hat{g} = 0$; the role of $\hat{g}$ on the buckling load is discussed in the Appendix.

Numerical Results and Concluding Remarks

The α, e, and η combinations for the axially stiffened cylinders that give $m = 1$ asymmetric mode (for the linear problem) for Z values in the range 10^2–10^4 were selected on the basis of Ref. 11. Three sets were chosen to roughly correspond to light, medium, and heavy stiffenings, as follows:

Table 1 Values of stiffening parameters

Stiffening	α_s	η_s	$\lvert e_s \rvert = \bar{E}_s/H$
Light	0.5	200	3
Medium	1.0	600	5
Heavy	1.5	1900	10

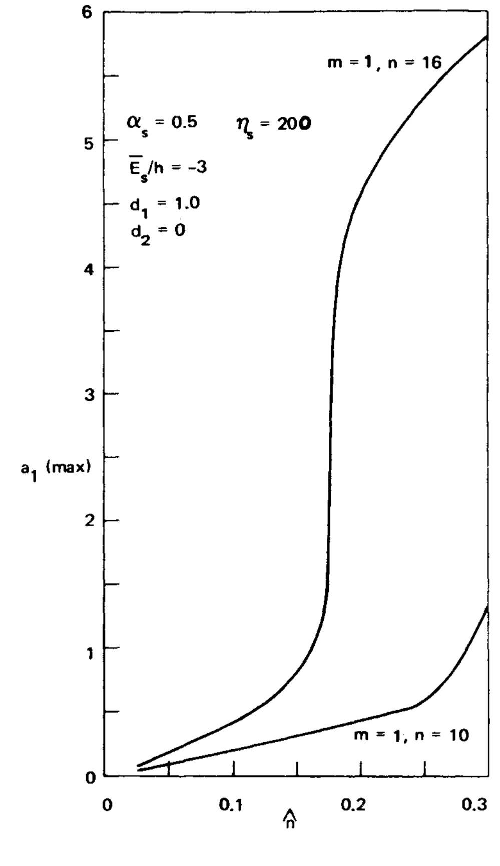

Fig. 3 Variation of a_1 max with $\hat{n}$.

Figure (3) shows a plot of a_1 max with load for a set of m, n values for a given imperfection parameter ($d_1 = 1$), and indicates the critical load (together with the corresponding m, n values) at which a well-defined jump in a_1 (max) occurs. The actual buckling load is the lowest of all possible critical for all m, n combinations. We found that for all the imperfect cylinders, with the properties according to Table 1, the lowest critical value was associated with $m = 1$ asymmetric mode, agreeing with the corresponding linear (perfect theory) problem.

Figures 4a–4c summarize the effects of eccentricity and imperfection on moderate-length axially-stiffened cylinders ($10^2 < Z < 10^4$) under axial step load for the three representative sets of parameters chosen.

These summary figures show that, for all the similarities with static analysis,[1,11] imperfect shells exhibit startlingly different behavior. Although the eccentricity effect is qualtitatively similar (cylinders with outside stiffeners buckling at higher loads than those with inside stiffeners) it is quantitatively different when imperfection is taken into account. For highly imperfect cylinders the eccentricity effect is not as pronounced as in the perfect cylinders, where a two-to-one reduction in buckling load is observed. This also implies that it is the cylinders with outside stiffeners that are relatively imperfection-sensitive. Hence the efficiencies claimed for outside stiffeners in aerospace usage[1] may have to be treated with some care.

A second important point is that even for relatively heavy stiffening systems Figs. 4b and 4c, the imperfection-sensitivity of axially stiffened cylinders (with outside stiffeners, to be sure) is still high. This confirms the suspicion raised by Hutchinson and Amazigo[5] regarding the imperfection sensitivity of axially stiffened cylinders, even though they were discussing the static behavior. It also confirms the conclusion of Budiansky[12] that imperfection-sensitive structures under static load will display the same property under step load.

Appendix

In deriving Eq. (25) from Eq. (18), if we set only a_i ($i = 2, 3, 4$) and d_1, d_2 equal to zero, then the term $(\hat{g}/2\alpha)$ a_1 will provide a linear term in a_1 equal to $-\frac{1}{2}(\nu n/1+\alpha_s)(a_1/\alpha)$. This term, when combined with the expression for $\hat{n}$ already in Eq. (18), will yield for the static value

$$\hat{n} = G_{rs} \Big/ \left(1 + \frac{\nu\beta^2}{1+\alpha_s}\right) \tag{A1}$$

which corresponds to the case of the static buckling of a cylinder under combined axial and lateral loads similar to the one treated by Lakshmikantham and Gerard.[13]

In order that the limiting case of zero imperfection go over to the corresponding static problem of an axially loaded cylinder, it is clear that $\hat{g}$ must be set equal to zero. Moreover the presence of $\hat{g}$ does not seem to affect the values for the imperfect case too much.

This point is elaborated upon somewhat, as it seems to be implied in Ref. 9, that the limiting solution (zero imperfection) is the classical static value for axially loaded cylinders independent of $\hat{g}$.

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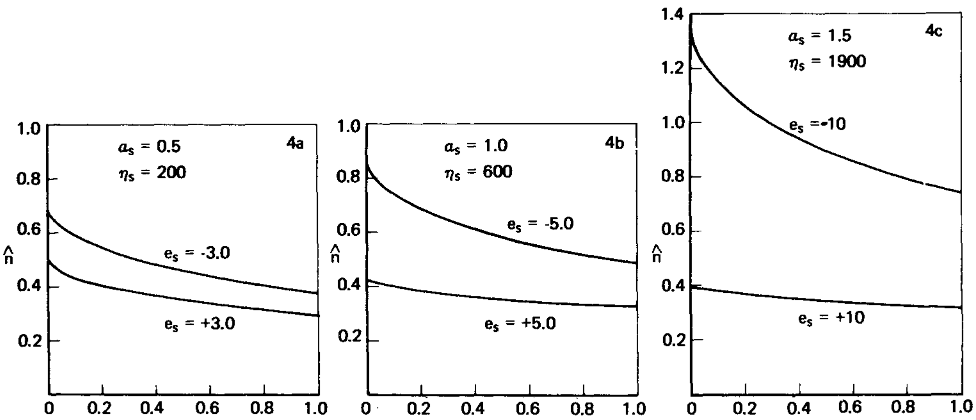

Fig. 4 Buckling load for an imperfect shell.

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